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Simple Practical Classical-H_2 Optimal Robust Controller

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I. Introduction

IN the control literature there are many results on linear-quadratic optimization and H_2 and H_∞ optimal control of linear systems with deterministic disturbances, Zhou and Doyle [1]. The corresponding optimal robust controllers have found numerous applications in aerospace engineering, see for example [2–6].

In this paper, we consider a simple practical linear-quadratic regulator problem with a Gaussian white noise stochastic disturbance and minimize the sum of a quadratic performance criterion and the H_2 norm of the closed-loop transfer function from the white noise disturbance to the system state space variables. It has been assumed that full state feedback is available for control purposes. The obtained results can be extended to the corresponding output feedback control problem with perfect state measurements as explained in remark 2.

Because the H_2 norm can be expressed in terms of the solution of the algebraic Lyapunov equation, it is shown that the solution to the above modern optimal control problem can be obtained by using a classic optimization technique based on the *matrix minimum principle* [7,8]. The necessary conditions for the optimal regulator gain lead to the problem of solving nonlinear coupled algebraic equations that have the structure of the generalized algebraic Lyapunov equations. These equations resemble the nonlinear equations of the classic output feedback optimal control problem [9]. Similarly, like for the output feedback equations [10,11], an algorithm is proposed for solving the obtained nonlinear algebraic equations by performing iterations on decoupled reduced-order algebraic Lyapunov equations. However, due to complexity of the nonlinear algebraic equations obtained further study is needed to clarify the issues of the existence of stabilizing solutions and to prove analytically the convergence of the proposed algorithm. An F-8 aircraft control system example under wind disturbances is included in the paper in order to demonstrate the proposed controller design procedure.

II. Problem Formulation

Consider the linear time invariant system with a Gaussian white noise stochastic disturbance

$$\dot{x}(t) = Ax(t) + Bu(t) + Gw(t) \tag{1}$$

where $x(t) \in \Re^n$ are state space variables, $u(t) \in \Re^m$ are control inputs, and $w(t) \in \Re^s$ are stationary Gaussian zero-mean white noise stochastic processes. The power spectral density matrix of the white noise stochastic process $w(t)$ is positive definite and denoted by W. The system state space variables are known to be stationary Gaussian stochastic processes whose statistics are completely described by their means and variances. The system output variables are assumed to be noise free (perfect measurements) and equal to the state variables (full state feedback).

The quadratic performance criterion to be minimized is given by

$$J_q = E\left(\lim_{t_f\to\infty}\left\{\frac{1}{t_f}\int_0^{t_f}[x^T(t)R_1x(t) + u^T(t)R_2u(t)]\,\mathrm{d}t\right\}\right) \tag{2}$$

with positive semidefinite R_1 and positive definite R_2. In the classic approach to the stochastic regulator problem with perfect measurements defined by (1) and (2), the optimal solution is given by [12]

$$u_c^*[x^*(t)] = -R_2^{-1}B^TKx^*(t) = -F_c^*x^*(t) \tag{3}$$

where $x^*(t)$ is the system trajectory under the optimal control $u^*[x^*(t)]$ and K satisfies the algebraic Riccati equation

$$A^TK + KA + R_1 - KBR_2^{-1}B^TK = 0 \tag{4}$$

The positive semidefinite stabilizing solution of (4) exists under the standard stabilizability–detectability conditions [12]. Note that the optimal solution to the linear-quadratic stochastic regulator problem with perfect measurements, as given by (3) and (4), is identical to the optimal solution of the linear-quadratic deterministic regulator problem.

The corresponding optimal performance criterion is given by

$$J_q^* = \mathrm{tr}\{(R_1 + F_c^{*^T}R_2F_c^*)\mathrm{var}(x^*)\} \tag{5}$$

where the state variance at steady state $\mathrm{var}(x^*)$ under the optimal control $u_c^*[x^*(t)]$, satisfies the algebraic Lyapunov equation

$$\mathrm{var}\,(x^*)(A - BF_c^*)^T + (A - BF_c^*)\mathrm{var}(x^*) + GWG^T = 0 \tag{6}$$

It should be pointed out that in the above optimization problem no attempt is made to minimize variances of the state space variables. Equation (6) just gives the corresponding variance matrix under the optimal control law (3) that minimizes (2).

In this paper, in addition to optimizing (2) along trajectories of (1), we optimize also the H_2 norm of the closed-loop transfer function from the white noise disturbance to the system output (in our case state variables). That will be done in the class of the linear feedback control laws

$$u[x(t)] = -Fx(t) \tag{7}$$

such that the sum of the quadratic performance criterion (2) and the H_2 norm of the closed-loop transfer function

$$G_w(s) = (sI - A + BF)^{-1}G, \qquad X(s) = G_w(s)W(s) \tag{8}$$

is minimized. It is known that the H_2 norm of (8) is given by [13]

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$$\|G_w(s)\|_2^2 = \mathrm{tr}\{P\} = J_w \tag{9}$$

where P satisfies the algebraic Lyapunov equation

$$(A - BF)P + P(A - BF)^T + GG^T = 0 = \mathcal{F}_1(P) \tag{10}$$

Note that J_w represents the impact of the white noise stochastic disturbance on the state space variables. By minimizing the disturbance impact through minimization of the H_2 norm of the closed-loop transfer function the variance of the state space variables will be reduced since by the Parseval's theorem an integral of the square of a time function is proportional to an integral of the magnitude square of the corresponding Laplace transform. Minimization of the variance of the state space variables is particularly important for aircraft during landing and takeoff, when the aircraft trajectories should be kept strictly on the designed paths with extremely small deviations allowed.

The criterion J_q under the linear feedback control law (7) can be expressed as [12]

$$J_q = \mathrm{tr}\{(R_1 + F^T R_2 F)Q\} \tag{11}$$

where Q satisfies

$$(A - BF)Q + Q(A - BF)^T + GWG^T = 0 = \mathcal{F}_2(Q) \tag{12}$$

The matrix Q represents the steady state variance of the system trajectories. Thus, one has to find a regulator gain that simultaneously minimizes (9) and (11) subject to constraints (10) and (12).

The above defined optimization problem can be converted into the following one:

$$\min_u J = \min_u (J_q + J_w) = \min_u \{\mathrm{tr}[P] + \mathrm{tr}[(R_1 + F^T R_2 F)Q]\} \tag{13}$$

subject to (10) and (12), which can be thought of as the Pareto game optimization problem with equal weights assigned to both performance criteria, [14]. It is straightforward to extend the considered optimization problem to the general Pareto game problem with

$$\min_u J = \min_u (\gamma_1 J_q + \gamma_2 J_w), \qquad \gamma_1 + \gamma_2 = 1, \qquad 0 \le \gamma_1, \gamma_2 \le 1$$

To solve the optimization problem defined by (10), (12), and (13), we use the matrix minimum principle [7].

III. Derivation of the Main Result

The Lagrangian of (10), (12), and (13) is given by

$$\mathcal{L} = \mathrm{tr}\{P + (R_1 + F^T R_2 F)Q\} + \mathrm{tr}\{[(A - BF)Q + Q(A - BF)^T + GWG^T]L^T\} + \mathrm{tr}\{[(A - BF)P + P(A - BF)^T + GG^T]H^T\} \tag{14}$$

Applying the necessary conditions for the optimum of the above static constrained optimization problem given by [7,8]

$$\frac{\partial \mathcal{L}}{\partial Q} = 0, \quad \frac{\partial \mathcal{L}}{\partial P} = 0, \quad \frac{\partial \mathcal{L}}{\partial F} = 0, \quad \frac{\partial \mathcal{L}}{\partial H} = 0, \quad \frac{\partial \mathcal{L}}{\partial L} = 0 \tag{15}$$

and using the known formulas for matrix trace derivatives

$$\frac{\partial}{\partial X}\mathrm{tr}\{MX\} = M^T, \qquad \frac{\partial}{\partial X}\mathrm{tr}\{MX^T\} = M$$
$$\frac{\partial}{\partial X}\mathrm{tr}\{MXN\} = M^T N^T, \qquad \frac{\partial}{\partial X}\mathrm{tr}\{MXNX^T\} = M^T X N^T + MXN \tag{16}$$

we get

$$\frac{\partial \mathcal{L}}{\partial Q} = 0 \Rightarrow (A - BF)^T L + L(A - BF) + R_1 + F^T R_2 F = 0 = \mathcal{F}_3(L) \tag{17}$$

$$\frac{\partial \mathcal{L}}{\partial P} = 0 \Rightarrow (A - BF)^T H + H(A - BF) + I = 0 = \mathcal{F}_4(H) \tag{18}$$

$$\frac{\partial \mathcal{L}}{\partial F} = 0 \Rightarrow R_2 F Q = B^T L Q + B^T H P \tag{19}$$

These equations together with Eqs. (10) and (12) comprise a set of necessary conditions for the minimum.

Under the assumption that $W > 0$ and by assuming that the gain F is stabilizing, the matrix Q obtained from (12) is positive definite so that the last equation can be written in the form

$$F = R_2^{-1} B^T L + R_2^{-1} B^T H P Q^{-1} = F_q + F_w \tag{20}$$

Note that the matrix L plays the role of the solution of the classic (standard) algebraic Riccati equation of the linear-quadratic (LQ) optimization and the second term in (20) indicates the gain change due to minimization of the H_2 norm of the closed-loop transfer function due to the white noise disturbance. Indeed, for $P = 0$, (17) and (20) produce the classic solution.

Remark 1: In a more general case the white noise intensity matrix W is positive semidefinite, in order words, white noise affects only some of the state variables. In that case, for the existence of the positive definite solution from the algebraic Lyapunov Eq. (12), one needs the additional assumption that the pair $[A, \mathrm{Chol}(W)]$ is observable; see Gajic and Qureshi [15], p. 7. Here, "Chol" stands for the Cholesky decomposition of the positive semidefinite matrix [16]. If this condition is not satisfied Eq. (19) has no unique solution with respect to F if such a solution exists at all.

Equations (10), (12), (17), (18), and (20) even though nonlinear coupled algebraic equations can be solved as decoupled algebraic Lyapunov equations by proposing an algorithm similar to those of the output feedback control problems [10,11]. The proposed algorithm is given below.

Algorithm:

Choose an initial condition for the gain $F^{(0)}$ such that the matrix $A - BF^{(0)}$ is asymptotically stable and solve iteratively

$$(A - BF^{(i)})P^{(i+1)} + P^{(i+1)}(A - BF^{(i)})^T + GG^T = 0 \tag{21}$$

$$(A - BF^{(i)})Q^{(i+1)} + Q^{(i+1)}(A - BF^{(i)})^T + GWG^T = 0 \tag{22}$$

$$(A - BF^{(i)})^T L^{(i+1)} + L^{(i+1)}(A - BF^{(i)}) + R_1 + F^{(i)^T} R_2 F^{(i)} = 0 \tag{23}$$

$$(A - BF^{(i)})^T H^{(i+1)} + H^{(i+1)}(A - BF^{(i)}) + I = 0 \tag{24}$$

$$F^{(i+1)} = R_2^{-1} B^T L^{(i+1)} + R_2^{-1} B^T H^{(i+1)} P^{(i+1)} Q^{(i+1)^{-1}} = F_q^{(i+1)} + F_w^{(i+1)} \tag{25}$$

The corresponding values for the approximate performance criterion at each iteration are obtained from

$$J^{(i)} = J_q^{(i)} + J_w^{(i)} = \mathrm{tr}[P^{(i)}] + \mathrm{tr}[(R_1 + F^{(i)^T} R_2 F^{(i)})Q^{(i)}] \tag{26}$$

Further study is needed in order to work out the complete convergence proof of the above algorithm as well as to establish analytically the existence of the stabilizing solutions of the nonlinear algebraic equations. This should be done along the results reported for the similar nonlinear algebraic equations coming from the static output feedback optimal control problem [10,11]. The Lyapunov iterations algorithm presented in this paper seems to be almost an universal tool for solving symmetric coupled nonlinear algebraic equations [15].

Remark 2: The results obtained can be extended to the output feedback control problem with perfect state measurement defined by

$$\dot{x}(t) = Ax(t) + Bu(t) + Cw(t)$$
$$y(t) = Cx(t)$$
$$u[x(t)] = -Fy(t) = -FCx(t) \tag{27}$$

Repeating the same optimization procedure, in this case, Eqs. (10), (12), and (17–19) respectively become

$$(A - BFC)P + P(A - BFC)^T + GG^T = 0 = \mathcal{F}_1^{\text{out}}(P) \tag{28}$$

$$(A - BFC)Q + Q(A - BFC)^T + GWG^T = 0 = \mathcal{F}_2^{\text{out}}(Q) \tag{29}$$

$$(A - BFC)^T L + L(A - BFC) + R_1 + C^T F^T R_2 FC = 0 = \mathcal{F}_3^{\text{out}}(L) \tag{30}$$

$$(A - BFC)^T H + H(A - BFC) + I = 0 = \mathcal{F}_4^{\text{out}}(H) \tag{31}$$

$$R_2 FCQ = B^T LQ + B^T HP \tag{32}$$

An algorithm in terms of Lyapunov iterations, similar to the previously proposed algorithm (21–25) for solving the corresponding full state optimal classical-H_2 robust control problem, can be proposed for solving algebraic Eqs. (28–32).

IV. Numerical Example

Consider an F-8 aircraft optimal linear-quadratic control problem under wind disturbances, Note that wind can be modeled as a white noise Gaussian stochastic process. The problem matrices taken from Teneketzis and Sandell [17] are given by

$$A = \begin{bmatrix} -0.01357 & -32.2 & -46.3 & 0 \\ 0.00012 & 0 & 1.214 & 0 \\ -0.0001212 & 0 & -1.214 & 1 \\ 0.00057 & 0 & -9.01 & -0.6696 \end{bmatrix}$$

$$B = \begin{bmatrix} -0.433 \\ 0.1394 \\ -0.1394 \\ -0.1577 \end{bmatrix}$$

$$G = [-46.3 \quad 1.214 \quad -1.214 \quad -9.01], \qquad W = 0.000315$$
$$R_1 = \text{diag}\{0.001, 0, 3260, 3260\}, \qquad R_2 = 3260$$

The corresponding state space variables and control input represent the following physical quantities:

$x_1(t)$: horizontal velocity deviation (ft/s),
$x_2(t)$: flight path angle (rad),
$x_3(t)$: angle of attack (rad),
$x_4(t)$: pitch rate (rad/s),
$u(t)$: elevator deflection (rad).

The optimal linear-quadratic control obtained from (3–6) produces

$$J_q = 24.9322, \qquad \text{tr}\{\text{var}(x^*)\} = 0.4447$$

Under the optimal control law, we have from (9) and (10) that $J_w = 1411.8$.

The proposed algorithm has produced the stabilizing solution with accuracy of $\mathcal{O}(10^{-6})$ after 17 iterations. For the initial gain we have used $F^{(0)} = F_c^*$. As a measure of accuracy we have taken

$$\min\{\text{norm}\mathcal{F}_1(P^{(17)}), \text{norm}\mathcal{F}_2(Q^{(17)}), \text{norm}\mathcal{F}_3(L^{(17)}), \text{norm}\mathcal{F}_4(H^{(17)})\}$$

where "norm" stands for the MATLAB function norm. Note that this is a pretty much conservative measure so that the results obtained are indeed better.

The following numerical results are obtained:

$$J_q^{(17)} = 36.9815, \qquad J_w^{(17)} = 653.1536, \qquad J^{(17)} = 690.1351$$

It is important to observe that the variance of the state variables is reduced in the case when the optimization (minimization) of the white noise transfer function is performed compared with the classic LQ stochastic optimization problem with perfect measurements. This can be seen from the results obtained from Eqs. (6) and (22), whose solutions produce

$$\text{tr}\{\text{var}(x)\} = \text{tr}\{Q^{(17)}\} = 0.2057 < \text{tr}\{\text{var}(x^*)\} = 0.4447$$

which indicates 53.7% variance reduction. The price for this reduction is paid by an increased value of the quadratic performance criterion from 24.93 to 36.98.

V. Conclusions

A simple practical linear feedback controller is proposed such that the sum of the quadratic performance criterion and the H_2 norm of the transfer function from white noise (disturbance) to the system state space variables is minimized. As a consequence of this, the variance of the system state variables is reduced as compared with the corresponding variance of the classic optimal stochastic regulator problem with perfect state measurements. The obtained results can be easily extended to the standard Pareto game optimization problem and, it is hoped, extended to the linear-quadratic Gaussian optimization problem with noisy measurements.

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